Inequality involving triangles

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Prove that for any acute triangle ABC the following inequality holds

$\frac{m_a}{h_a}\cos A + \frac{m_b}{h_b}\cos B + \frac{m_c}{h_c}\cos C \geq \frac{3}{2}$.

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Let F be area of the triangle. Then $\sum \frac{m_a}{h_a}\cos A = \sum \frac{am_a}{2F}\cos A$ and, therefore,

$\sum \frac{m_a}{h_a}\cos A \geq \frac{3}{2} \Leftrightarrow \sum am_a \cos A \geq 3F$.

Since* $\frac{b^2+c^2}{4R} \leq m_a, abc = 4RF$ and $b\cos A + a\cos B = c$ we obtain that

$\sum am_a\cos A \geq \sum a \bullet \frac{b^2+c^2}{4R}\cos A = \frac{1}{4R}\sum ab(b\cos A + a\cos B) = \frac{3abc}{4R} = 3F$.

* Proof of inequality $\frac{b^2+c^2}{4R} \leq m_a$.

Let R and d_a be, respectively, circumradius and distance from the circumcenter to side a. Then by triangle inequality $|m_a - R| \leq d_a$ and, since

$d_a = \sqrt{R^2 - \frac{a^2}{4}}$ we obtain $|m_a - R| \leq \sqrt{R^2 - \frac{a^2}{4}} \Leftrightarrow m_a^2 - 2m_aR + R^2 \leq R^2 - \frac{a^2}{4} \Leftrightarrow$

$4m_a^2 - 8m_aR + a^2 \leq 0 \Leftrightarrow 2(b^2+c^2) - a^2 - 8m_aR + a^2 \leq 0 \Leftrightarrow b^2 + c^2 \leq 4m_aR$.